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Cosmological D-instantons and cyclic universes

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Abstract

For models of gravity coupled to hyperbolic sigma models, such as the metric-scalar sector of IIB supergravity, we show how smooth trajectories in the ‘augmented target space’ connect FLRW cosmologies to non-extremal D-instantons through a cosmological singularity. In particular, we find closed cyclic universes that undergo an endless sequence of big-bang to big-crunch cycles separated by instanton ‘phases’. We also find ‘big-bounce’ universes in which a collapsing closed universe bounces off its cosmological singularity to become an open expanding universe.

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1. Introduction

Homogeneous and isotropic cosmological solutions of gravity coupled to N scalar fields can be viewed as trajectories in a Lorentzian-signature ‘augmented target space’ of dimension $N + 1$. If the scalar field target space is the hyperbolic space H_N , it can happen that the augmented target space is a Milne universe, and that cosmological singularities correspond to points on trajectories at which the Milne horizon is crossed [1]. In the models considered in [1], cosmological trajectories were geodesics, and hence straight lines in a Milne patch of Minkowski ‘spacetime’⁵. In such models the Milne horizon is typically crossed twice, corresponding *either* to expansion from a big-bang singularity followed by contraction to a big-crunch singularity, *or* to a big-bang/big-crunch transition through a region behind the Milne horizon in which the scale factor is imaginary. As the trajectory through the horizon is smooth in Minkowski coordinates, this construction strongly suggests an interpretation in

⁵ See [2] for a related observation in the context of flat $k = 0$ cosmologies of three-dimensional gravity. It was shown in [3] that cosmological trajectories are *always* geodesics in an appropriate metric on the augmented target space but it is only in rather special cases that this metric is the Milne metric on Minkowski spacetime.

which a collapsing universe tunnels through a ‘forbidden’ region in field space to emerge as a re-expanding universe [1].

The main aim of this paper is to provide further support for this idea by showing, in a particular class of models, that the trajectory behind the Milne horizon corresponds to a solution of the field equations of the *Euclidean* action; in other words, an instanton. Of course, the general idea that a big-bang singularity might be resolved by a transition to a Euclidean signature solution, or gravitational instanton, is not new. Here, however, we are not actually resolving the singularity in the spacetime metric; we are instead re-interpreting it as a mere coordinate singularity in a larger space in which the scalar fields are on the same footing as the metric. The instanton solutions that we need for this re-interpretation are also different; they are the ‘super-extremal’ D-instantons [4] that generalize the (extremal) D-instanton of Euclidean IIB supergravity [5], hence our title. An amusing by-product of our analysis is that for $k = 1$ cosmologies in a certain subclass of models, these instanton–cosmology transitions may link up to yield a *cyclic* universe; i.e. one that expands from a big bang to a big crunch, passes through an instanton ‘phase’ to re-emerge as an expanding universe that again recollapses to a big crunch, followed by a further instanton ‘phase’, *ad infinitum*. Another subclass of models, which include IIB supergravity, allow ‘big-bounce’ universes that simply bounce smoothly off the big-crunch singularity without the need for an instanton ‘phase’.

We shall simplify our task by taking $N = 2$ but we consider an arbitrary, but *finite*, radius of the scalar field target space. We shall also keep arbitrary the spacetime dimension d . The model to be considered here has zero scalar potential, so our starting point (in the conventions of [4]) is the Lagrangian density

$$\mathcal{L} = \sqrt{\epsilon \det g} \left[R - \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\epsilon e^{b\phi}(\partial\chi)^2 \right], \quad (1.1)$$

for d -metric g and scalar fields (ϕ, χ) . For $\epsilon = -1$, the scalar fields parametrize a hyperbolic space H_2 of radius $2/b$. For $d = 10$ and $b = 2$ we then have a consistent truncation of IIB supergravity; note that in this context, χ is a pseudo-scalar so that a solution with non-zero χ breaks parity. For the same values of d and b the Lagrangian with $\epsilon = 1$ is the consistent truncation of the Euclidean IIB supergravity shown in [5, 4] to admit D-instanton solutions. Note that the scalar fields in this Euclidean action parametrize the Lorentzian-signature space adS_2 , so the Euclidean action is not positive definite, but can be made positive definite [6] by dualization of χ (followed by the usual ‘conformal rotation’ of Euclidean quantum gravity [7]).

The above Lagrangian density is thus a natural generalization to arbitrary spacetime dimension and arbitrary target space radius of the metric-scalar sector of IIB supergravity, or its Euclidean counterpart. As we shall soon see, cosmological trajectories of this model correspond to motion in a Milne universe. It was observed in [1] that there is a special value of the target space radius (and hence of b) for which the motion is geodesic. However, as we show here, exact cosmological and instanton solutions can be found for any target space radius.

2. Cosmologies and instantons

To investigate cosmological solutions of our model, or to find instanton solutions of its Euclidean version, we make the ansatz

$$ds^2 = \epsilon(e^{\alpha\phi}f)^2 d\lambda^2 + e^{2\alpha\phi/(d-1)} d\Sigma_k^2, \quad \phi = \phi(\lambda), \quad \chi = \chi(\lambda), \quad (2.1)$$

where f is an arbitrary function of λ , and

$$\alpha = \sqrt{\frac{d-1}{2(d-2)}}. \quad (2.2)$$

The $(d-1)$ -metric $d\Sigma_k^2$ is (at least locally) a maximally symmetric space of positive ($k=1$), negative ($k=-1$) or zero ($k=0$) curvature. One can choose coordinates such that

$$d\Sigma_k^2 = (1 - kr^2)^{-1} dr^2 + r^2 d\Omega_{d-2}^2, \quad (2.3)$$

where $d\Omega_{d-2}^2$ is an $SO(d-1)$ -invariant metric on the unit $(d-2)$ -sphere. This ansatz constitutes a consistent reduction of the original degrees of freedom to a three-dimensional subspace, the ‘augmented target space’, with coordinates (φ, ϕ, χ) . The full equations of motion reduce to a set of equations that can themselves be derived by variation of the time-reparametrization invariant effective action

$$I = \frac{1}{2} \int d\lambda \{ f^{-1} (\epsilon \dot{\varphi}^2 - \epsilon \dot{\phi}^2 + e^{b\phi} \dot{\chi}^2) + 2k(d-1)(d-2)f e^{\varphi/\alpha} \}, \quad (2.4)$$

where the overdot indicates differentiation with respect to λ . For $\epsilon = -1$ we can interpret λ as a time coordinate related to the time t of FLRW cosmology in standard coordinates by

$$dt \propto e^{\alpha\varphi} f d\lambda. \quad (2.5)$$

For $\epsilon = 1$ the metric has Euclidean signature and we can interpret λ as imaginary time.

Before proceeding, it is convenient to define new scalar field variables (ψ, θ) by

$$\begin{aligned} e^{(b/2)\phi} &= e^\psi \cos^2(\theta/2) - \epsilon e^{-\psi} \sin^2(\theta/2), \\ e^{(b/2)\phi} \chi &= b^{-1} (e^\psi + \epsilon e^{-\psi}) \sin \theta, \end{aligned} \quad (2.6)$$

to get the new effective action

$$I = \frac{1}{2} \int d\lambda \left\{ \frac{4}{b^2} f^{-1} \left[\frac{b^2}{4} \epsilon \dot{\varphi}^2 - \epsilon \dot{\psi}^2 + \frac{1}{4} (e^\psi + \epsilon e^{-\psi})^2 \dot{\theta}^2 \right] + 2k(d-1)(d-2)f e^{\varphi/\alpha} \right\}. \quad (2.7)$$

This is just a reparametrization of the target space but it has the advantage that the new coordinates are globally valid. Introducing the new scale-factor variable η by

$$\eta^\gamma = 2\gamma(d-1) e^{\varphi/(2\alpha)}, \quad (2.8)$$

where

$$\gamma = 1/(b\alpha), \quad (2.9)$$

we arrive at the action

$$I = \frac{1}{2} \int d\lambda \left\{ \frac{4}{b^2} f^{-1} \left[\epsilon (\dot{\eta}/\eta)^2 - \epsilon \dot{\psi}^2 + \frac{1}{4} (e^\psi + \epsilon e^{-\psi})^2 \dot{\theta}^2 \right] + \frac{b^2}{4} k f \eta^{2\gamma} \right\}. \quad (2.10)$$

We remark, for future reference, that the ansatz (2.1) leads to $\gamma = 2/3$ for $d = 10$ IIB supergravity.

Because of the time-reparametrization invariance, we are free to choose the function f ; each choice of f corresponds to some choice of time parameter. There are two choices that are particularly convenient, and we now consider them in turn.

2.1. The ‘Liouville’ gauge

The simplest way to proceed for general b is to make the gauge choice

$$f = 4/b^2. \quad (2.11)$$

From (2.10) one sees that the effective Lagrangian in this gauge is

$$L = \frac{1}{2}[-\epsilon \dot{\psi}^2 + \frac{1}{4}(\mathrm{e}^\psi + \epsilon \mathrm{e}^{-\psi})^2 \dot{\theta}^2] + \frac{1}{2}[\epsilon(\dot{\eta}/\eta)^2 + k\eta^{2\gamma}]. \quad (2.12)$$

Apart from the constraint, the dynamics of the motion on the target space, which is manifestly geodesic, is now separated from the dynamics of the scale factor, which is determined by a equation of Liouville type; for this reason we will call this choice of gauge the ‘Liouville gauge’.

As $Sl(2; \mathbb{R})$ is the isometry group of both H_2 (the target space of the Lorentzian action) and adS_2 (the target space of the Euclidean action), there is a conserved $Sl(2; \mathbb{R})$ ‘angular momentum’ ℓ^μ , and the geodesics are such that

$$\dot{\psi}^2 - \epsilon \frac{1}{4}(\mathrm{e}^\psi + \epsilon \mathrm{e}^{-\psi})^2 \dot{\theta}^2 = \ell^2. \quad (2.13)$$

The constraint (f equation of motion) is

$$(\dot{\eta}/\eta)^2 = \ell^2 + k\epsilon\eta^{2\gamma}. \quad (2.14)$$

We now present the solutions of the equations of motion of (2.12) subject to the constraints (2.13) and (2.14), first for the target space fields and then for the scale factor.

2.1.1. Target space geodesics. Geodesics on the H_2 ($\epsilon = -1$) or adS_2 ($\epsilon = 1$) target space are solutions of the field equations of (2.12) for ψ and θ subject to (2.13) and can be classified as follows, according to whether ℓ^2 is positive, negative or zero:

- $\ell^2 > 0$. For $\epsilon = 1$ the solution is

$$\begin{aligned} \sinh \psi &= \pm \sqrt{1 + \frac{q_-^2}{\ell^2}} \sinh[\ell(\lambda - \lambda_0)] \\ \tan(\theta - \theta_0) &= \pm \frac{q_-}{\ell} \tanh[\ell(\lambda - \lambda_0)], \end{aligned} \quad (2.15)$$

for constants λ_0, θ_0 and q_- (this being the integration constant for the super-extremal D-instanton of [4]). For $\epsilon = -1$ the solution is

$$\begin{aligned} \cosh \psi &= \sqrt{1 + \frac{q_-^2}{\ell^2}} \cosh[\ell(\lambda - \lambda_0)] \\ \tan(\theta - \theta_0) &= \pm \frac{q_-}{\ell} \coth[\ell(\lambda - \lambda_0)]. \end{aligned} \quad (2.16)$$

In the special case that $q_- = 0$ these solutions simplify, for either choice of the sign ϵ , to

$$\psi = \pm \ell(\lambda - \lambda_0), \quad \theta = \theta_0, \quad (\epsilon = \pm 1). \quad (2.17)$$

- $\ell^2 < 0$. In this case only $\epsilon = 1$ is possible, and the solution is

$$\begin{aligned} \sinh \psi &= \pm \sqrt{\frac{q_-^2}{(-\ell^2)} - 1} \sin[\sqrt{-\ell^2}(\lambda - \lambda_0)] \\ \tan(\theta - \theta_0) &= \pm \frac{q_-}{\sqrt{-\ell^2}} \tan[\sqrt{-\ell^2}(\lambda - \lambda_0)]. \end{aligned} \quad (2.18)$$

- $\ell^2 = 0$. The only solution for $\epsilon = -1$ in this case is the trivial one for which both ψ and θ are constant. For $\epsilon = 1$ the solution is

$$\sinh \psi = \pm q_-(\lambda - \lambda_0), \quad \tan(\theta - \theta_0) = \pm q_-(\lambda - \lambda_0). \quad (2.19)$$

It should be noted that, in each case, the \pm signs for ψ and θ can be chosen independently.

2.1.2. *The scale factor.* We next turn to the constraint (2.14). Given ℓ^2 , this determines η as follows:

- $\ell^2 > 0$.

$$\eta^{2\gamma} = \eta_0^{2\gamma} \exp(\pm 2\ell\gamma\lambda), \quad (k = 0), \quad (2.20)$$

$$\eta^{2\gamma} = \frac{\ell^2}{\sinh^2(\ell\gamma\lambda)}, \quad (k\epsilon = 1), \quad (2.21)$$

$$\eta^{2\gamma} = \frac{\ell^2}{\cosh^2(\ell\gamma\lambda)}, \quad (k\epsilon = -1), \quad (2.22)$$

for some constant η_0 . Note that all $k = \pm 1$ trajectories are asymptotic to some $k = 0$ trajectory near $\eta = 0$, as expected since the σ -model matter satisfies the strong energy condition.

- $\ell^2 < 0$. In this case, there is a solution only for $k = \epsilon = 1$:

$$\eta^{2\gamma} = \frac{-\ell^2}{\sin^2(\gamma\sqrt{-\ell^2}\lambda)}, \quad (k = \epsilon = 1). \quad (2.23)$$

- $\ell^2 = 0$. In this case there is a solution only for $k\epsilon \geq 0$. If $k = 0$ then η is constant. Otherwise

$$\eta^{2\gamma} = 1/(\gamma\lambda)^2, \quad (k\epsilon = 1). \quad (2.24)$$

For $\epsilon = k = 1$ these solutions yield the super-extremal ($\ell^2 > 0$), sub-extremal ($\ell^2 < 0$) and extremal ($\ell^2 = 0$) D-instantons of [4]. For $\epsilon = -1$ they yield FLRW cosmologies; from (2.5) we see that the standard FLRW time t is related to the parameter λ by

$$dt \propto \eta^{2\gamma\alpha^2} d\lambda. \quad (2.25)$$

Given one of the above solutions for $\eta^{2\gamma}$ as a function of λ we can determine λ as a function of t and hence η as a function of t . Of most interest here is the behaviour near $\eta = 0$. For example, for $\ell^2 > 0$ we have

$$\eta \sim \eta_0 e^{-\ell\lambda}, \quad (2.26)$$

for $\lambda \rightarrow \infty$, as $\eta \rightarrow 0$. This yields (for a choice of integration constant such that $t \rightarrow 0$ as $\lambda \rightarrow \infty$)

$$-t \propto e^{-2\gamma\alpha^2\ell\lambda}. \quad (2.27)$$

Given that we start with a cosmological solution for negative t , this shows that a big-crunch singularity will be approached as $t \rightarrow 0$. By considering the behaviour as $\lambda \rightarrow -\infty$ we may similarly deduce that a cosmological solution for positive t must have had a big-bang singularity at $t = 0$. In other words, cosmologies with $\ell^2 > 0$ are incomplete in the sense that they have a beginning or an end (or both) at finite t . We shall see in the following section how they can be completed.

2.2. The ‘Milne’ gauge

Returning to (2.10), we make the new gauge choice

$$f = \frac{4}{b^2 \eta^2}. \quad (2.28)$$

As the possible choices of f are related by a redefinition of the independent variable, we will need to distinguish the independent variable in this gauge from the parameter λ used previously. Let us call the new independent variable τ ; it is related to λ through the differential equation

$$d\tau = \eta^2(\lambda) d\lambda, \quad (2.29)$$

which can be solved for $\tau(\lambda)$ given any of the scale-factor solutions $\eta(\lambda)$ presented above.

In the gauge (2.28), the action is

$$I = \int d\tau L_\tau, \quad (2.30)$$

where⁶

$$L_\tau = \frac{1}{2}\epsilon \left(\frac{d\eta}{d\tau} \right)^2 + \frac{1}{2}\eta^2 \left[-\epsilon \left(\frac{d\psi}{d\tau} \right)^2 + \frac{1}{4}(\epsilon^\psi + \epsilon \epsilon^{-\psi})^2 \left(\frac{d\theta}{d\tau} \right)^2 \right] + \frac{k}{2}\eta^{2\gamma-2}. \quad (2.31)$$

We observe that for $\epsilon = -1$ the kinetic term is that of a particle in a three-dimensional Milne ‘universe’, so we call this choice of gauge the ‘Milne’ gauge.

The Milne universe is actually just Minkowski space in an analogue of spherical polar coordinates. The Cartesian field variables X_μ ($\mu = 0, 1, 2$) are

$$\begin{aligned} X_0 &= \pm \frac{1}{2}\eta(\epsilon^\psi - \epsilon \epsilon^{-\psi}) \\ X_1 &= \pm \frac{1}{2}\eta(\epsilon^\psi + \epsilon \epsilon^{-\psi}) \cos \theta \\ X_2 &= \pm \frac{1}{2}\eta(\epsilon^\psi + \epsilon \epsilon^{-\psi}) \sin \theta. \end{aligned} \quad (2.32)$$

Note that

$$X^2 \equiv -X_0^2 + X_1^2 + X_2^2 = \epsilon \eta^2. \quad (2.33)$$

Since η^2 is positive, it follows that $X^2 < 0$ when $\epsilon = -1$, and $X^2 > 0$ when $\epsilon = 1$. The $X^2 < 0$ region is the Milne region of Minkowski space and cosmological solutions are trajectories in this space. Generic trajectories reach $\eta = 0$ at finite FLRW time, corresponding to a cosmological singularity. However, the hypersurface $\eta = 0$ is just the Milne horizon, and the singularity at the Milne horizon disappears in the Cartesian coordinates X_μ . The trajectory can therefore be smoothly continued through the Milne horizon *in Cartesian coordinates* to the region in which $X^2 > 0$, where we need $\epsilon = 1$. Thus, on passing through the Milne horizon, a cosmological trajectory becomes an instanton (and vice versa).

The target space and the scale-factor solutions given previously can now be combined into a single solution for X_μ . For example, for $\ell^2 > 0$, the solutions are

$$X_\mu = \begin{cases} \pm \eta[s_\mu \sinh(\ell\lambda) + c_\mu \cosh(\ell\lambda)], & \epsilon = 1, \\ \pm \eta[s_\mu \cosh(\ell\lambda) + c_\mu \sinh(\ell\lambda)], & \epsilon = -1, \end{cases} \quad (2.34)$$

⁶ Note that $L_\tau d\tau = L d\lambda$, where L is the Lagrangian in the gauge used previously.

where

$$\begin{aligned}
 s_0 &= \sqrt{1+a^2} \cosh(\ell\lambda_0), & a &\equiv q_-/\ell, \\
 c_0 &= -\sqrt{1+a^2} \sinh(\ell\lambda_0), \\
 c_1 &= \cosh(\ell\lambda_0) \cos(\theta_0) + a \sinh(\ell\lambda_0) \sin(\theta_0), \\
 s_1 &= -\sinh(\ell\lambda_0) \cos(\theta_0) - a \cosh(\ell\lambda_0) \sin(\theta_0), \\
 c_2 &= -a \sinh(\ell\lambda_0) \cos(\theta_0) + \cosh(\ell\lambda_0) \sin(\theta_0), \\
 s_2 &= a \cosh(\ell\lambda_0) \cos(\theta_0) - \sinh(\ell\lambda_0) \sin(\theta_0).
 \end{aligned} \tag{2.35}$$

Note that $(c_\mu \pm s_\mu)$ is null.

3. Instanton-cosmology transitions

The Milne gauge Lagrangian L_τ in Cartesian coordinates is

$$L_\tau = \frac{1}{2}[(dX/d\tau)^2 + k(\epsilon X^2)^{\gamma-1}]. \tag{3.1}$$

The constraint is now

$$(dX/d\tau)^2 = k(\epsilon X^2)^{\gamma-1}. \tag{3.2}$$

We thus have a problem analogous to that of a particle of zero energy in a central potential, with conserved $SL(2; \mathbb{R})$ ‘angular momentum’

$$\ell^\mu = \epsilon^{\mu\nu\rho} X_\nu (dX_\rho/d\tau). \tag{3.3}$$

In contrast to the usual central potential problem, X^2 can be zero (or negative) for non-zero 3-vector X and ℓ^2 may also be positive, negative or zero. Nevertheless, the problem is still exactly soluble; the solutions are the solutions $X_\mu(\lambda)$ given earlier with λ expressed as a function of τ . However, if $\eta \rightarrow 0$ as $|\lambda| \rightarrow \infty$ it is generally necessary to piece together several of the previously given solutions to obtain a full trajectory in the 3D Minkowski space. In this section, we shall show how this can be done.

The formulation of the problem in terms of Cartesian field variables X_μ makes it obvious how trajectories can be smoothly continued through the Milne horizon in certain special cases. For example, if γ is an odd integer then the Lagrangian (3.1) is independent of ϵ . It then follows from standard theorems about the existence and uniqueness of solutions of ordinary differential equations that any trajectory that crosses $\eta = 0$ will connect smoothly to a solution on the other side of the Milne horizon with the same value of k ; a simple example is $\gamma = 1$, for which the trajectories are straight lines. We discuss this case in detail below. If γ is an even integer then the Lagrangian (3.1) depends on both ϵ and k but only through the combination ϵk . It follows that any trajectory that crosses $\eta = 0$ must smoothly join to a solution on the other side of the Milne horizon with a flipped sign of k . A simple example is $\gamma = 2$, which we also discuss in detail below.

It may not be obvious that a flip of the sign of k is consistent with continuity in the full field space, prior to imposing the ansatz (2.1), because one might expect the $k = 1$ and $k = -1$ metrics to belong to disjoint subspaces in the space of all metrics. However, as previously observed, the $k = \pm 1$ trajectories all approach a $k = 0$ trajectory near $X^2 = 0$, so the actual radius of curvature goes to infinity at a cosmological singularity whatever the value of k . In other words, the subspace of $k = 1$ FLRW metrics is joined to the subspace of $k = -1$ FLRW metrics precisely at the points in the full space of fields at which we flip the sign of k , so there is no discontinuity caused by this sign change.

Another simple case which we analyse in detail below is $k = 0$, in which case the results are obviously γ -independent⁷.

3.1. The $k = 0$ case: flat cosmologies

For $k = 0$, the equation of motion for X^μ is trivially solved by

$$X_\mu = a_\mu + p_\mu \tau, \quad (3.4)$$

for constant 3-vectors a and p . The constraint implies that $p^2 = 0$; in other words, the trajectories in Minkowski space are null. In this case,

$$\ell^2 = (a \cdot p)^2, \quad (3.5)$$

and it follows that $X^2 = a^2 \pm 2\ell\tau$. If $\ell \neq 0$ we are free to shift τ to put this into the form

$$X^2 = \pm 2\ell\tau. \quad (3.6)$$

Integrating (2.29) then yields

$$\tau \propto e^{\pm 2\ell\lambda}, \quad (3.7)$$

and therefore

$$\eta^2 \equiv \epsilon X^2 \propto e^{\pm 2\ell\lambda}. \quad (3.8)$$

Note that

$$t \propto \pm \tau, \quad (3.9)$$

whereas $\tau(\lambda) > 0$. It follows, given a choice of sign, that the solution (2.20) covers only the part of the trajectory for which t is either positive or negative, but not both. Any null geodesic must cross the horizon once, at $t = 0$, so if we associate $t > 0$ with a big-bang cosmology then we must associate $t < 0$ with a pre-big-bang instanton. These solutions give rise to the upper and lower diagonal lines in figure 1; the two possibilities correspond to the sign choice in (3.8). The transition from instanton to cosmology, or vice versa, occurs at the hypersurface $X^2 = 0$, which becomes the hyperbola $X_0^2 - X_1^2 = X_2^2$ when projected onto the (X_0, X_1) -plane.

The remaining possibility for $k = 0$ is to have $a \cdot p = 0$, i.e. $\ell^2 = 0$. In this case, $X^2 = a^2$ and the geodesic will never reach the Milne horizon. This corresponds to the middle diagonal line in figure 1.

Thus, for $k = 0$ we have instanton–cosmology transitions with $\ell^2 > 0$ for any value of γ . In particular, this is true for IIB supergravity (for which $\gamma = 2/3$).

3.2. The $\gamma = 1$ case: geodesics

The Lagrangian L_τ is especially simple for $\gamma = 1$; in this case, the equation of motion for X_μ is solved by

$$X_\mu = a_\mu + p_\mu \tau, \quad (3.10)$$

⁷ There is one other circumstance in which the physics is independent of γ : it is obvious from (1.1) that solutions with $\chi \equiv 0$ cannot depend on b and hence that solutions with $\theta \equiv 0$ cannot depend on γ . These are the solutions with $X_2 = 0$. It is not immediately obvious from the Lagrangian (3.1) why this should be the case, but this can be seen from an application of the Jacobi principle (see, e.g., [1]), which states that zero-energy solutions of the equations of motion of (3.1) are geodesics in the metric $(\epsilon X^2)^{\gamma-1} dX \cdot dX$, which is a conformal rescaling of the 3D Minkowski metric. On the 2D subspace with $X_2 = 0$, we can introduce lightcone coordinates X_\pm and write the metric as $(\epsilon X_- X_+)^{\gamma-1} dX_- dX_+$. Setting $U = (\epsilon X_-)^{\gamma/2}$, $V = (\epsilon X_+)^{\gamma/2}$, the metric becomes a constant times $dU dV$, for any γ .

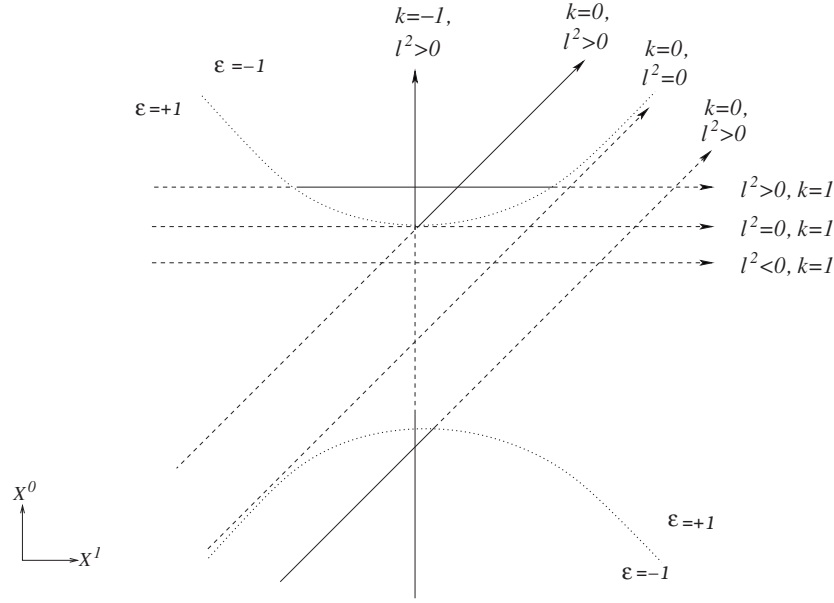


Figure 1. Instantons and cosmologies as geodesics in a projection of Milne spacetime. The solid (dashed) lines are cosmologies (instantons), which are separated by the dotted hyperbola with $X_0^2 - X_1^2 = X_2^2$.

for constant 3-vectors a and p , and the constraint implies that $p^2 = k$. Note that

$$\ell^2 = (a \cdot p)^2 - ka^2. \quad (3.11)$$

As long as $k \neq 0$ we can shift τ , if necessary, to arrange for $p \cdot a = 0$, in which case $a^2 = -k\ell^2$ and

$$X^2 = k(\tau^2 - \ell^2). \quad (3.12)$$

Note, however, that $p \cdot a = 0$ implies that ℓ is non-spacelike if p is timelike; i.e., $\ell^2 \geq 0$ if $k = -1$.

The $\ell^2 > 0$ cases are especially interesting. Consider first the $k = -1$ subcase, for which X is timelike for $|\tau| > \ell$ but spacelike for $|\tau| < \ell$. In other words, a single straight-line solution in Minkowski space can be viewed as a cosmology ($\epsilon = -1$) for $|\tau| > \ell$ but as an instanton ($\epsilon = 1$) for $|\tau| < \ell$. As argued in [1] (in the context of another model with similar features), this corresponds to a big-crunch/big-bang transition through a classically forbidden region behind the Milne horizon. For $k = 1$ the roles of cosmology and instanton are reversed, and the cosmological region (in which $X^2 < 0$) corresponds to a universe expanding from a big bang (where the trajectory first crosses the Milne horizon) to a big crunch (where it again crosses the Milne horizon). For each of these two subcases, (2.29) reduces to

$$d\lambda = \frac{k\epsilon d\tau}{\tau^2 - \ell^2}, \quad (\ell^2 > 0), \quad (3.13)$$

and we must solve this for $\tau(\lambda)$ in two cases:

- $k\epsilon = -1$, $|\tau| < \ell$. In this case,

$$\tau = \ell \tanh(\ell\lambda), \quad (3.14)$$

and hence

$$\eta^2 = X^2 = \frac{\ell^2}{\cosh^2(\ell\lambda)}. \quad (3.15)$$

This is the $\gamma = 1$ case of solution (2.21).

- $k\epsilon = 1$. $|\tau| > \ell$. In this case,

$$\tau = -\ell \coth(\ell\lambda), \quad (3.16)$$

and hence

$$\eta^2 = -X^2 = \frac{\ell^2}{\sinh^2(\ell\lambda)}. \quad (3.17)$$

This is the $\gamma = 1$ case of solution (2.22).

Thus, in these cases the full trajectory connects a collapsing FLRW universe to an expanding FLRW universe via an instanton solution; in fact, to one of the super-extremal D-instantons of [4]. As noted above, a change of sign of k reverses the roles of cosmology and instanton, so which of the two ($k = \pm 1$) super-extremal D-instantons is relevant depends on the sign of k . Both these $\ell^2 > 0$ possibilities are illustrated in figure 1, where the timelike geodesics with $k = -1$ are vertical lines while the spacelike geodesics with $k = +1$ are horizontal.

A similar calculation for $\ell^2 < 0$, for which we must set $\epsilon = k = 1$ for a non-trivial solution, yields

$$\tau(\lambda) = -\sqrt{-\ell^2} \cot(\sqrt{-\ell^2}\lambda), \quad (3.18)$$

and thence

$$\eta^2 = \frac{-\ell^2}{\sin^2(\sqrt{-\ell^2}\lambda)}. \quad (3.19)$$

This is the sub-extremal D-instanton solution (2.23), which is *periodic* in imaginary time.

Finally, we consider $\ell^2 = 0$ (which includes $\ell_\mu = 0$, and hence $a_\mu = 0$, as a subcase). In this case, $X^2 = k\tau^2$ and hence $\tau = -(\epsilon k)/\lambda$. This yields $\eta^2 \equiv \epsilon X^2 = (\epsilon k)/\lambda^2$, which implies both $\epsilon k = 1$ and $\eta^2 = 1/\lambda^2$. This is the $\gamma = 1$ case of solution (2.24). However, this solution must be interpreted *either* as a cosmology (when $k = -1$) *or* as an instanton, in fact the extremal D-instanton (when $k = 1$) *but not both*. For the interpretation as a $k = -1$ cosmology we noted previously that this case yields a d -dimensional Milne universe⁸ for which the apparent cosmological singularity is actually just a coordinate singularity that can be resolved without the need for scalar fields. Thus, we should not expect (and have not found) any ‘instanton phase’ of the cosmological trajectory in this case. The cosmological and instanton interpretations are, in this case, *separate* solutions, not linked by a spacetime cosmological singularity. In figure 1, we have illustrated the $k = 1$ instanton with $\ell^2 = 0$. The corresponding cosmological solution with $k = -1$ is not indicated but consists of the $\ell \rightarrow 0$ limit of the $\ell^2 > 0$ case. This limit implies that $X_2 \rightarrow 0$: the two hyperbolae join to form a cross and the ‘instanton phase’ of the solution disappears.

We have now seen how *all* the solutions found in section 2 correspond, for $\gamma = 1$, to some straight-line trajectory in the three-dimensional Minkowski spacetime with coordinates X_μ . Note, in particular, that a single straight-line trajectory in Minkowski ‘superspace’ corresponds to cosmological and instanton solutions with the *same* value of k .

⁸ This is a *spacetime* Milne universe, and is not to be confused with the Milne ‘universe’ in which cosmological trajectories evolve.

3.3. The $\gamma = 2$ case: cyclic universes

There is one other value of γ for which the equations we have to solve are *linear* in Minkowski coordinates, namely $\gamma = 2$. We see from (3.1) that the effective Lagrangian in this case is

$$L_\tau = \frac{1}{2}[\dot{X}^2 + (k\epsilon)X^2]. \quad (3.20)$$

The equations of motion are

$$\ddot{X}_\mu = (k\epsilon)X_\mu, \quad (\mu = 0, 1, 2). \quad (3.21)$$

We must choose solutions that satisfy the constraint

$$\dot{X}^2 = (k\epsilon)X^2, \quad (3.22)$$

which can be interpreted as a ‘zero-energy’ condition. We have already discussed the $k = 0$ case, so we may assume that $k \neq 0$. We now consider in turn the two possible values of $k\epsilon$:

- $k\epsilon = 1$. In this case, the equations are solved by

$$X_\mu = A_\mu e^\tau + B_\mu e^{-\tau}, \quad (3.23)$$

for real 3-vectors A, B , and the constraint implies that

$$A \cdot B = 0. \quad (3.24)$$

This implies that

$$X^2 = A^2 e^{2\tau} + B^2 e^{-2\tau}, \quad (3.25)$$

and that

$$\ell^2 = -4A^2 B^2. \quad (3.26)$$

If both A and B are null then we have a solution with $\ell^2 = 0$ and $X^2 = 0$. This does *not* correspond to one of the solutions found in section 2 because the coordinates used there do not cover the hypersurface $X^2 = 0$, but this ‘extra’ solution is of no physical interest. For the other $\ell^2 = 0$ cases, we may assume without loss of generality that only A is null, so that

$$X^2 = B^2 e^{-2\tau}, \quad (3.27)$$

where B is non-null. If A is non-zero then B must be spacelike in order to be orthogonal to A , in which case $X^2 > 0$. This case is relevant only for $\epsilon = 1$, in which case $k = 1$. These are extremal D-instantons for $\gamma = 2$. If $A = 0$ then B may be either spacelike or timelike. If B is spacelike then we must choose $\epsilon = 1$ and we then have a further special case of the extremal D-instanton. If B is timelike then we must choose $\epsilon = -1$ and we then have the unique ($k = -1$) $\ell^2 = 0$ FLRW cosmology, for which $\eta \propto e^{-\tau}$. This is non-zero for all finite τ , so it might appear that this is a cosmology without a singularity. However, as we shall see in the following section, the $\gamma = 2$ case is a very special one for $\ell^2 = 0$ because τ becomes infinite for finite FLRW time t .

This leaves those cases for which neither A nor B is null. If both are timelike or spacelike then $\ell^2 < 0$ and X^2 is never zero. If one is timelike and the other spacelike then $\ell^2 > 0$ and X^2 passes through zero; this is case (2.21) of subsection 2.1.2.

- $k\epsilon = -1$. In this case,

$$X_\mu = C_\mu e^{i\tau} + \bar{C}_\mu e^{-i\tau}, \quad (3.28)$$

where C_μ is a complex 3-vector (with complex conjugate \bar{C}_μ) subject to the constraint

$$C \cdot \bar{C} = 0. \quad (3.29)$$

This implies that

$$X^2 = C^2 e^{2i\tau} + \bar{C}^2 e^{-2i\tau}, \quad (3.30)$$

and that

$$\ell^2 = |2C^2|^2 \geq 0. \quad (3.31)$$

For $\ell^2 = 0$ (which occurs when C is null) we again have the $X^2 = 0$ case. Otherwise, $\ell^2 > 0$ and X^2 passes through zero whenever

$$\tau = \tau_0 + n\pi/2, \quad (n \in \mathbb{Z}), \quad (3.32)$$

where τ_0 is any solution of $e^{4i\tau} = -\bar{C}^2/C^2$.

As anticipated, all non-trivial cosmological trajectories with $\ell^2 > 0$ have an instanton phase, behind the Milne horizon, to which they are smoothly connected through a cosmological singularity. These trajectories connect $k = \pm 1$ cosmologies with $k = \mp 1$ instantons, and in the $k\epsilon = -1$ case the trajectories are cyclic universes. We conclude this subsection with an explicit example.

First, note that the constraint (3.29) is solved, without loss of generality, by

$$C_\mu = A(1, e^{i\beta_1} \cos \xi, e^{i\beta_2} \sin \xi), \quad (3.33)$$

where A is a real constant determined by $|2C^2| = \ell$, and β_1, β_2, ξ are three real constant angles. This yields

$$X_\mu = 2A(\cos \tau, \cos(\tau + \beta_1) \cos \xi, \cos(\tau + \beta_2) \sin \xi). \quad (3.34)$$

Consider the particular case $\beta_1 = -\frac{\pi}{2}, \beta_2 = 0$, for which

$$X_\mu = \frac{\sqrt{\ell}}{\cos \xi}(\cos \tau, \sin \tau \cos \xi, \cos \tau \sin \xi). \quad (3.35)$$

In this case $X^2 = -\ell \cos 2\tau$, independent of ξ . Recalling that $\epsilon X^2 \geq 0$ for all τ , we see that

$$\epsilon X^2 = \ell |\cos 2\tau|. \quad (3.36)$$

Using this in $d\tau = \eta^2 d\lambda$, we may integrate to deduce that $|\cos 2\tau| = 1/\cosh 2\ell\lambda$, and hence that

$$\epsilon X^2 = \frac{\ell}{\cosh 2\ell\lambda}. \quad (3.37)$$

This is precisely the $\gamma = 2$ case of (2.21), where for $\epsilon = -1$ we viewed it as a big-bang to big-crunch $k = 1$ cosmology. For this interpretation, we should choose $\tau \in (-\pi/4, \pi/4)$ but we have now discovered that we can follow this solution through the big crunch at $\tau = \pi/4$ to an instanton solution, given by the same formula but with $\tau \in (\pi/4, 3\pi/4)$. This then re-emerges as another big-bang cosmology for $\tau \in (3\pi/4, 5\pi/4)$. If the first one evolved in the future-Milne region of the Minkowski ‘superspace’ then the second evolves in the past-Milne region. After a further recollapse to an instanton phase for $\tau \in (5\pi/4, 7\pi/4)$, the cycle is completed as this gives way to a new big-bang universe. There are thus two big bang and two big-crunch singularities in every cycle, but the transition through them is smooth in the augmented target space (see figure 3).

3.4. Generic γ

As already observed, the Lagrangian (3.1) tells us that for general γ we have a problem analogous to that of a particle in a central potential. Using (2.32) to return to ‘polar’ coordinates, we find that the Lagrangian governing the radial motion is

$$L_{\text{rad}} = \frac{\epsilon}{2} \left[\left(\frac{d\eta}{d\tau} \right)^2 + \frac{\ell^2}{\eta^2} + (k\epsilon)\eta^{2(\gamma-1)} \right]. \quad (3.38)$$

As expected, we have a ‘centrifugal’ term for non-zero ℓ^2 . Recall that the constraint implies that we must retain only the ‘zero-energy’ solutions of the equation of motion, for which

$$\left(\frac{d\eta}{d\tau} \right)^2 = \frac{\ell^2}{\eta^2} + (k\epsilon)\eta^{2(\gamma-1)}. \quad (3.39)$$

This is equivalent to (2.14) but with τ as the independent variable. The solutions are therefore the same as those given in 2.1.2 but with η expressed as a function of τ rather than λ .

For $\ell^2 < 0$, we have a centrifugal barrier that prevents η from passing through zero. For $\ell^2 > 0$, the centrifugal barrier becomes a ‘centrifugal well’ that dominates near $\eta = 0$. For $k\epsilon = -1$, the potential becomes positive for sufficiently large η ; as positive values of the potential are not accessible for zero energy, the ‘particle’ is confined to finite η , and must fall to $\eta = 0$ because the potential has no stationary points (under the assumed conditions). For $\ell^2 > 0$ and $k\epsilon = 1$, the potential is always negative; it has a stationary point if $\gamma > 1$ but the energy is fixed at zero so all trajectories must start at infinite η and then fall to $\eta = 0$. The net conclusion is that η reaches zero on *all* $\ell^2 > 0$ trajectories, in agreement with our Cartesian coordinate analyses for $\gamma = 1$ and $\gamma = 2$. Moreover, from the explicit $\ell^2 > 0$ solutions (2.20)–(2.22), and the relation $\eta^2 d\lambda = d\tau$, one may show that

$$\eta^2 \propto |\tau| \propto |t|^{1/(\gamma\alpha^2)}, \quad \text{as } \eta \rightarrow 0, \quad (\ell^2 > 0), \quad (3.40)$$

where t is the FLRW time, and here we have chosen the time origin such that $t \rightarrow 0$ as $\tau \rightarrow 0$. This shows that there is a cosmological singularity at $t = 0$, corresponding to the time $\tau = 0$ at which the cosmological trajectory reaches $\eta = 0$.

The $\ell^2 = 0$ case is special and needs a separate discussion. In this case, we may assume that $k\epsilon = 1$ because otherwise there is no solution other than the trivial $k = 0$ solution for which η is constant. Given $k\epsilon = 1$, there is a negative effective potential and hence no obvious barrier to prevent η passing through zero. However, the $\ell^2 = 0$ solution is

$$\eta = \begin{cases} [(\gamma - 2)\tau]^{1/(2-\gamma)}, & \gamma \neq 2, \\ e^{-\tau}, & \gamma = 2. \end{cases} \quad (3.41)$$

This shows that η reaches zero *at finite* τ if $\gamma < 2$ and *at infinite* τ if $\gamma \geq 2$ (in agreement with our earlier analysis of the $\gamma = 1$ and $\gamma = 2$ cases). This does not necessarily mean that η will not pass through zero if $\gamma \geq 2$ because it may happen that τ becomes infinite for finite FLRW time t . To see whether this happens we need to consider the relation between t and τ , which is

$$dt = \eta^{2(\gamma\alpha^2-1)} d\tau. \quad (3.42)$$

This yields for all values of γ

$$\eta = \left(\frac{\gamma t}{2-d} \right)^{(d-2)/\gamma}, \quad (\ell^2 = 0). \quad (3.43)$$

One sees from this result that for all values of γ there is a cosmological singularity at $t = 0$, i.e. *at finite FLRW time*. However, as already noted for $\gamma = 1$, there is no transition from a cosmology to an instanton, or vice versa, when $\ell^2 = 0$.

For $\gamma = 1$ and $\gamma = 2$, we have seen that in Minkowski field variables there is actually a smooth connection at $\eta = 0$ onto an instanton solution in the region behind the Milne horizon. We now want to determine whether a similar smooth transition is possible for other values of γ . As we have just seen, this issue arises only for $\ell^2 > 0$, so we now restrict our discussion to that case. From the general solutions found previously we see that a trajectory can reach $X^2 = 0$ only as $|\lambda| = \infty$. Let us concentrate on the case in which $\eta \rightarrow 0$ as $\lambda \rightarrow \infty$; in this case⁹

$$\lim_{\lambda \rightarrow \infty} X_\mu(\lambda) = c_\mu + s_\mu, \quad (3.44)$$

independently of k . It follows that the $\epsilon = -1$ solution for X_μ can be matched continuously at $\lambda = \infty$ onto the $\epsilon = 1$ solution at $\lambda = \infty$, *without changing parameters*; i.e., with the same λ_0 , q_- and θ_0 for the instanton and cosmological solution.

The next question is whether the transition is continuous for the first derivatives. Using the formulae given above, we find the following results¹⁰:

$$\lim_{\lambda \rightarrow \infty} \frac{dX_i}{dX_0} = \begin{cases} (c_i + s_i)/(c_0 + s_0) & \gamma < 1 \\ c_i/c_0 & \gamma = 1, k = 1 \\ s_i/s_0 & \gamma = 1, k = -1 \\ (c_i - s_i)/(c_0 - s_0) & \gamma > 1 \end{cases} \quad (i = 1, 2). \quad (3.45)$$

Thus, the first derivatives match in all cases with the same choice of θ_0 , q_- and λ_0 parameters on both sides.

Note that for $\gamma = 1$ continuity of the first derivatives implies that one must patch together solutions with the same value of k , leading to the straight lines of section 3.1. For all other values of γ , we could choose to patch together solutions with the same or opposite k , but continuity of second derivatives imposes further restrictions. From the general solutions given previously, one can show that

$$\lim_{\lambda \rightarrow \infty} \frac{d^2 X_i}{dX_0^2} = \begin{cases} 0 & \gamma < 1/2 \\ \epsilon \times \text{const} & \gamma = 1/2 \\ \epsilon \times \infty & \frac{1}{2} < \gamma < 1 \\ 0 & \gamma = 1 \\ \epsilon k \times \infty & 1 < \gamma < 2 \\ \epsilon k \times \text{const} & \gamma = 2 \\ 0 & \gamma > 2 \end{cases} \quad (i = 1, 2). \quad (3.46)$$

This shows, for instance, that when $1 < \gamma \leq 2$ the second derivatives are continuous if $k\epsilon$ is the same on both sides, meaning that on a smooth trajectory k must flip sign across the Milne horizon; otherwise there is a discontinuity in the second derivatives. For $\gamma > 2$, the second derivative is continuous irrespective of whether k flips sign; moreover, as already observed, there is an analytic extension for integer γ . For $\gamma < 1/2$, there is also no discontinuity in second derivatives whether or not k flips sign. However, when $\frac{1}{2} \leq \gamma < 1$, there is a discontinuity in second derivatives irrespective of whether k flips or remains the same, which means that there is no smooth transition through the Milne horizon for γ in this range.

To summarize, there are transitions between instanton and cosmological ‘phases’ for $k \neq 0$ that are continuous up to and including second derivatives provided that $\gamma < 1/2$ or $\gamma \geq 1$. For example, a $k = 1$ universe, which would normally be thought to start with a big

⁹ Recall that the parameter λ in the $X^2 > 0$ region is independent of the parameter λ in either of the $X^2 < 0$ regions.

¹⁰ The γ -dependence arises because the (γ -independent) leading terms at large λ of $\frac{dX_\mu}{d\lambda}$ cancel.

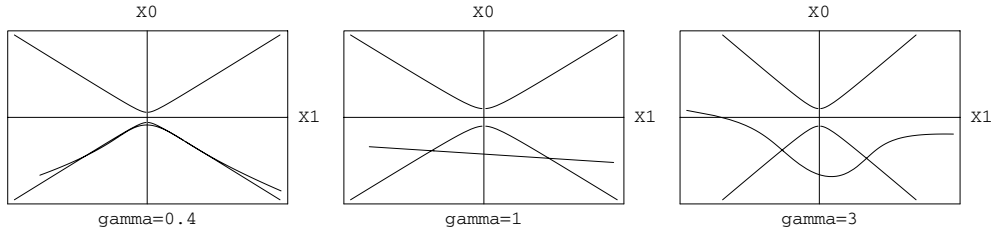


Figure 2. Instanton extensions of closed big-bang to big-crunch universes for $\gamma = 0.4$, $\gamma = 1$ and $\gamma = 3$. The $\gamma = 1$ curve is a straight line and the $\gamma = 3$ curve is analytic.

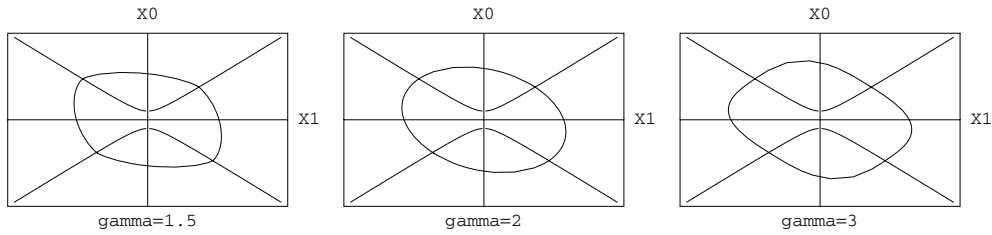


Figure 3. Cyclic universes with $k\epsilon = -1$ for $\gamma > 1$. The generic $\gamma = 2$ curve is an ellipse.

bang and end with a big crunch, may actually be part of a larger instanton–cosmology with pre-big-bang and post-big-crunch instanton phases, as illustrated in figure 2 for $\gamma = 0.4$, $\gamma = 1$ and $\gamma = 3$. The $\gamma = 0.4$ curve is probably not analytic. In general, we would expect continuity in all derivatives to impose further restrictions, and it may be that an analytic continuation is possible only for integer γ .

As we have seen for the $\gamma = 2$ case, these instanton–cosmology transitions can connect to form cyclic universes, and this remains true for all even γ ; in fact, it is true for all $\gamma > 1$ if one requires only continuity up to and including second derivatives. This is illustrated in figure 3.

3.5. Big-bounce universes for IIB supergravity

There is an important feature that distinguishes the trajectories in $\gamma < 1$ models from those in $\gamma > 1$ models. In both cases, a trajectory that approaches the Milne horizon is null at the point of crossing. For $\gamma < 1$, this null curve approaches a null geodesic generator of the cone $X^2 = 0$, since $dX^\mu \propto X^\mu$. In contrast, for $\gamma > 1$, dX^μ is not proportional to X^μ .

This means that there is an additional possibility for $\gamma < 1$: in this case, a cosmological ($\epsilon = -1$) trajectory with $k = \pm 1$ may be smoothly joined to another cosmological ($\epsilon = -1$) solution $k = \mp 1$, where by ‘smooth’ we mean continuity up to and including second derivatives.

For example, a collapsing closed universe can bounce off its big-crunch singularity to begin another phase as an expanding open universe. This possibility is illustrated in figure 4 for $\gamma = 2/3$.

This possibility is relevant to IIB supergravity since $\gamma = 2/3$ for the uncompactified $d = 10$ theory. By considering a Kaluza–Klein ansatz of the form

$$ds^2 = ds_d^2 + dy_n^2, \quad \phi = \phi_d, \quad \chi = \chi_d, \quad (n = d + 1, \dots, 10), \quad (3.47)$$

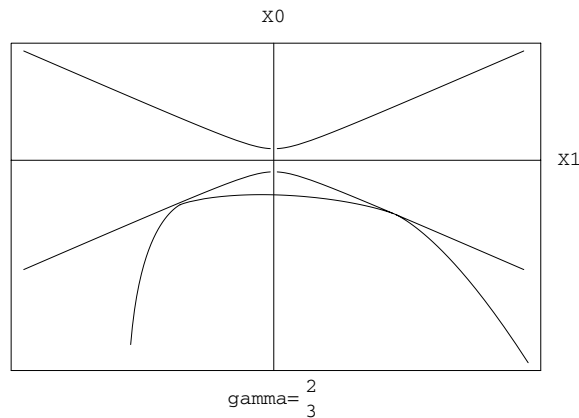


Figure 4. A big-bounce universe for $\gamma = 2/3$, in which an open universe that is collapsing to a big crunch bounces off the Milne horizon in the augmented target space to emerge as a closed big-bang universe. The latter necessarily collapses to a big-crunch singularity, but this is again another bounce off the Milne horizon from which there emerges an open, but expanding, universe.

we get an effective action in $d < 10$ dimensions with¹¹

$$\gamma = \frac{1}{2} \sqrt{\frac{2(d-2)}{(d-1)}}. \quad (3.48)$$

Assuming that $d \geq 3$, this implies that $\gamma \in [1/2, 2/3]$. This is precisely in the range $\frac{1}{2} \leq \gamma < 1$ for which a smooth transition to an instanton is not possible (for $k \neq 0$). However, a smooth bounce *is* possible, as we have just seen.

4. Discussion

In this paper, we have expanded on the observation in [1] that cosmological singularities may be resolved in models with scalar fields parametrizing a hyperbolic target space via an interpretation as coordinate singularities of a Milne ‘super-metric’ on the ‘augmented target space’ of scalar fields *and* metric scale factor. Specifically, the segments of cosmological trajectories that lie behind the Milne horizon (for which the scale factor of the Lorentzian-signature spacetime is imaginary) have been shown to correspond, in a particular class of models with H_2 target space, to the ‘super-extremal’ D-instanton solutions of [4].

It is interesting that the original, extremal, D-instanton of [5], generalized to d spacetime dimensions, does not connect to a cosmological solution in the way that the super-extremal D-instanton does, and the same is true of the sub-extremal instantons of [4]. The reason for this is that the extremal and sub-extremal instantons are non-singular, so there is no way to join them on to any cosmological solution in a continuous way. Recalling the debate over the significance of singularities in the context of the Hawking–Turok cosmological instanton [9], it is amusing to note that singularities of D-instantons are essential to the cosmological interpretation that we have proposed for them.

¹¹ By embedding the lower-dimensional scalars differently one can obtain other values of γ , however. For example, $N = 8$ supergravity in four dimensions can be truncated to gravity coupled to a vector and a scalar with dilaton couplings $a = 0, 1/\sqrt{3}, 1, \sqrt{3}$ [8]. Upon reduction to $d = 3$ these give rise to our model (1.1) with $\gamma = 1, \sqrt{3}/2, 1/\sqrt{2}, 1/2$. Note that these are all in the range $1/2 \leq \gamma \leq 1$.

The models considered here generalize the metric-scalar sector of IIB supergravity to arbitrary spacetime dimension d and an arbitrary H_2 radius, which is simply related to the parameter γ arising in our analysis. The $\gamma = 1$ case is particularly simple as the cosmological-instanton trajectories are just straight lines. In particular, the $k = -1$ universes of this model undergo a big-crunch/big-bang transition of the type proposed in [1], for which we have here identified the intermediate ‘instanton phase’. Another special case is $\gamma = 2$, for which closed universes allow a smooth continuation to cyclic universes corresponding to closed curves in the analytic extension of the augmented target space. The idea that the universe may be cyclic is an old one that has recently been revived in the braneworld approach to cosmology [10]; here we have found an explicit model that realizes this idea.

For $\gamma < 1/2$ and $\gamma \geq 1$, corresponding to two disjoint ranges of the H_2 radius, we have shown that there exists a continuation of cosmological trajectories through the Milne horizon with continuous first and second derivatives, and the continuation is analytic for integer γ . For the values $1/2 \leq \gamma < 1$, which are relevant for IIB supergravity and its compactifications to $d < 10$, we found that there is a smooth ‘big-bounce’ solution in which a collapsing cosmology is bounced off the big crunch to become an expanding big-bang universe.

Our results are also of potential relevance to IIA string theory because in the special case of zero axion, i.e. $\chi = 0$, the Lagrangian (1.1) is a truncation of massless IIA supergravity, or its Euclidean counterpart. Interestingly, there is an instanton/cosmology solution that survives this truncation, which corresponds to the hypersurface $X_2 = 0$ in Cartesian coordinates; it is just the $q_- = 0$ solution (2.17) with the scale factor η given by equations (2.20)–(2.22). For $d = 10$ and $\epsilon = k = 1$, this is the non-extremal IIA D-instanton of [11]; it has an M-theory origin because the Kaluza–Klein ansatz

$$ds_{11}^2 = e^{2\alpha\phi/(d-1)} ds_{10}^2 - \epsilon e^{-\phi/\alpha} dx_{10}^2 \quad (4.1)$$

takes the $d = 11$ Einstein–Hilbert Lagrangian to the zero axion truncation of (1.1) for $d = 10$. One can thus show that both ‘phases’ of the IIA instanton/cosmology solution have a common M-theory origin as a Schwarzschild black hole: the IIA instanton is obtained by reducing the black-hole exterior spacetime over time [11, 4] while the IIA cosmology is obtained by reducing the interior spacetime over a space direction [12]. An interesting open question is whether there is an extension to the massive IIA theory, for which there is a scalar potential for the dilaton. We hope to show in a future paper how the results obtained here generalize to models with a scalar potential that include as special cases both massive IIA supergravity and the model with cosmological constant that was used in [1] to study the big-crunch to big-bang transition for flat cosmologies.

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